The Math Less Traveled

(working title)

Chapter 2: Numbers
draft version 1
June 24, 2007

This chapter traces the concept of *number* from the naturals to the reals, while more generally exploring the ideas of *generalization* and *deduction* through this particular lens.

Chapter 2

Numbers

What is a number?

Does this question seem silly or simplistic to you? Is the answer 'obvious'? . . . Don't be so sure. How, exactly, would you explain to someone else what a number is? Perhaps you might say that a number is something you use to count. However, that definition breaks down quickly. What what about zero? Can you 'count' zero things? How about negative numbers, or $3/4$, or $\sqrt{2}$? Are those numbers? What about (gasp) i?

The fact is that there's no nice, neat definition which can tell us what counts as a number and what doesn't. Partly this is because disagreements over this very question have been historically common among mathematicians. More importantly, the modern concept of number is fundamentally based in analogy, rather than an arbitrary set of predefined rules. Simply put, something is a number if it has certain things in common with other kinds of numbers.[1] This makes the topic of numbers a perfect place to see some of the most important processes in mathematics: *generalization* and *deduction*. Generalization (also known as *abstraction*) is about extracting the essence of an idea in order to gain a deeper or broader understanding. "What is the underlying pattern here? Can I come up with a way to de-

[1] This seems circular, but it isn't, since we can start by agreeing on the counting numbers $(1, 2, 3 \ldots)$ being numbers, not because they're like other sorts of numbers, but just because.

scribe more things, or cover more cases?" Deduction is about starting with a set of premises, and following them to their logical conclusions: "If I start with *this*, where does it logically take me?"

In this chapter we'll trace the development of the concept of *number*,[2] starting from the very basics and working our way up to a more modern conception.

And before you conclude that this will be a boring chapter to be endured so we can get on to the interesting topics—numbers have a funny way of concealing much more than meets the eye. You may think that you already know everything in this chapter . . . but you just might be surprised.[3]

2.1 Millions of mammoths

We'll start in the same place that prehistoric hunter-gatherer societies probably started when they first hit upon the idea of counting things:

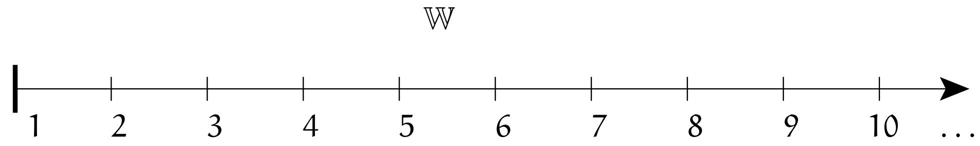
$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots$$

One day a prehistoric hunter-gatherer-person came to the insightful realization that a herd of six mammoths and a pile of six rocks have something in common—their 'six-ness'—and the concept of *number* was born! This seems painfully obvious to us, but the fact is that there are primitive languages which don't have general number-words, but use different words for things like "ten coconuts" and "ten boats" (Conway and Guy 1996).

As an aside, those prehistoric hunter-gatherer-people probably just thought of anything more than 10 (or so) as "a lot," but to us the dots mean something slightly different (we'll see what in a moment).

[2] Have you noticed that the word *number* starts to sound really weird if you say it a lot? Number number number number number.

[3] Or perhaps you won't. If after reading the whole chapter you find that you weren't surprised, feel free to skip directly to the next chapter.



These numbers are usually called *whole numbers*, and can be represented by the symbol $\mathbb{W}$. Sometimes you might also hear them called *counting numbers*. We can define them very simply in a way that mimics counting: 1 is a whole number; and if n is a whole number, then so is $n + 1$. So that means 2 is a whole number. And if 2 is a whole number, then so is 3. Which means that 4 is a whole number. And the fact that 4 is a whole number means . . . and so on.

Problem 2.1 ($\star$). What is the biggest whole number you can think of? Write it here: __________________________. ★

Ha ha! Good joke, huh? You know you're a math nerd if you laughed. Which is nothing to be ashamed of, by the way! Math is totally sweet and there's no use pretending it isn't. If you didn't laugh, don't worry, there's hope for you yet.

The point, of course, is that there are *infinitely many* whole numbers—they keep on going forever.

2.2 Addition

Getting back to our prehistoric friends,[4] the first thing they would have thought to do with these 'whole numbers' they invented is to add them. If you have eight mammoth carcasses, and someone gives you three more (a very thoughtful gift!), how many do you have?[5]

Addition has several nice properties with fancy names: it is *commutative* (you can add two numbers in either order, that is, $a+b$ always equals $b+a$) and *associative* (if you have three numbers to add, it doesn't matter which

[4] Let's call them Og and Ig.
[5] A lot.

two you add first; that is, $(a+b)+c$ always equals $a+(b+c)$). We'll have more to say about these properties in Chapter ??.

Problem 2.2 ($\star$). What is $2+3+8+7+6+4+5+9+5+1+23+7$? Can you do it in your head? Fast? ★

Our whole numbers have another nice property. Try this: pick any two whole numbers. Go on, pick two! Any two whole numbers you like. Got your numbers? Okay. Now add them together. Now, do you have the answer? I want you to concentrate on the answer very hard. Hmm . . . let's see . . . concentrate a little harder, please . . . hmm . . . I think . . . yes, I'm pretty sure . . . aha! I'll bet you $3.14 that the answer you're thinking of is . . . another whole number![6] I'm right, aren't I? That's a good trick to try on your friends sometime.

OK, seriously, though, the fact that you always get another whole number when adding two whole numbers is very important. The mathematical way to say this is that the whole numbers are *closed under addition*. This is a great property for a set of numbers to have with respect to a particular operation, because it means there is one less thing to worry about; you can never 'fall off the edge of the world,' so to speak, when performing the operation.[7] In fact, you'll see that much of our journey through different sorts of numbers in this chapter will be motivated by a desire to make our set of numbers closed once again after introducing some new operation.

2.3 A section about nothing

Our whole numbers are great, but they're missing something. Actually, more to the point, they're missing nothing. A number to represent nothing, that is. To the rescue comes zero! Zero, of course, is how many things you have when you don't have anything. It's the number of pages of this book

[6] Please try to contain your amazement.

[7] The difference between having a set of numbers which is closed under a particular operation and one which isn't is sort of like the difference between running around blindfolded in a big grassy field or on the edge of the Grand Canyon.

which contain color photographs of weasels. It's how many mammoths you killed when you got chased home by a saber-toothed tiger and never even saw a mammoth.[8] Zero is also the *identity with respect to addition*: adding zero has no effect whatsoever. In other words, 0 is the number for which

$$0 + a = a + 0 = a, \tag{2.1}$$

for any number a.

Our new set of numbers—whole numbers along with zero—is often called the set of *natural numbers*, represented by the symbol $\mathbb{N}$.[9] It shouldn't be too hard to convince yourself that addition of natural numbers is still commutative and associative, and that the natural numbers are closed under addition, just as the whole numbers were.[10]

2.4 Multiplication

The second most obvious operation to perform using natural numbers is multiplication. If you kill three mammoths a day for a whole week, how many have you killed?[11] Multiplication is simply a shorthand for repeated addition:

$$a \cdot b = \underbrace{b + b + b + \cdots + b}_{a}. \tag{2.2}$$

In other words, a times b means to add b to itself a times.[12] Note that

[8] Although it should be pointed out that our prehistoric friends probably didn't think of it that way; it took a while for the idea of zero *as a number* to catch on.

[9] Note, however, there is no widespread agreement on whether $\mathbb{N}$ should include zero or not. Some people use the term *natural numbers* to refer to what we have called *whole numbers*. There's no right or wrong answer for this particular question; you just have to make sure you always know which definition is being used.

[10] Did you flip through the book to see if there are any color photographs of weasels? You did, didn't you? Come on, don't you trust me?

[11] You guessed it—a lot. You would also be very tired.

[12] A word on notation: in elementary school everyone learns to use the symbol $\times$ to represent multiplication ($a \times b$), but mathematicians in general prefer to use a centered dot ($a \cdot b$) instead, or more often still, no symbol at all (ab). Mathematicians probably use

this definition works even when $a = 0$: in that case, we add together zero copies of b, and adding up nothing gives us 0. Let's establish a few basic properties of multiplication.

Problem 2.3 ($\star$). What is the identity with respect to multiplication?

Problem 2.4 ($\star\star$). Prove that multiplication is commutative (that is, that $a \cdot b = b \cdot a$ for any natural numbers a and b). ★

Problem 2.5 ($\star\star$). Prove that multiplication is associative (that is, that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any natural numbers a, b, and c.

Problem 2.6 ($\star$). Are the natural numbers closed under multiplication?

So far, we see that multiplication has a lot of the same properties as addition. There's one important property left that has to do with the way multiplication and addition interact, the *distributive property*. The distributive property says that if a, b, and c are natural numbers, then

$$a \cdot (b + c) = a \cdot b + a \cdot c. \tag{2.3}$$

We say that multiplication *distributes over* addition.

Problem 2.7 ($\star\star$). Prove the distributive property. ★

Problem 2.8 ($\star$). Does addition distribute over multiplication? Why or why not? ★

2.5 There is no such thing as subtraction

Say what? No such thing as subtraction? Preposterous! Everyone knows there is too such a thing as subtraction; it's the opposite of addition. That is,

$$c - b = a \iff a + b = c. \tag{2.4}$$

a centered dot or nothing because it is easy to write and multiplication is such a frequent operation; elementary school students are probably taught to use $\times$ since it would be too easy for them to confuse $2 \cdot 4$ with 2.4. But you're big enough to handle the truth. By the way, when is $a \cdot b = a.b$?

For example, $7 - 4 = 3$, since $3 + 4 = 7$. As another example, $2 - 5 =$ Aieeeeeeeeeeeeee!

Oops.

Obviously, the natural numbers are not closed under subtraction! We just 'fell off the edge of the world' since there's no *natural number* n for which $n + 5 = 2$. Ah, you say, but all we have to do in order to fix things is introduce negative numbers. Well, you're right—but stop and think for a moment about what a strange and abstract idea it is to have numbers which are less than zero! We're so used to them that they seem natural, even obvious. But it hasn't always been that way—in fact, it took Western mathematicians quite a long time to fully accept the idea of negative numbers. It's true that we can connect negative numbers to the 'real world' by thinking of them as representing debts or losses, but a 'debt' is still much more abstract than a pile of rocks!

Now that we have a proper appreciation of how crazy it is to have negative numbers, let's define them. For every number[13] n, we'll define *negative* n (written $-n$) to be the *additive inverse* of n. That is, $-n$ is the number which you can add to n in order to get zero, the additive identity:

$$n + (-n) = 0. \tag{2.5}$$

Problem 2.9 ($\star$). Show that $-0 = 0$, and $-1 \neq 1$.

Problem 2.10 ($\star$). Everyone knows that $5 + (-3) = 2$. Can you prove it using equation (2.5)? ★

Problem 2.11 ($\star$). We know that -7 is the additive inverse of 7. What is the additive inverse of -7? Can you prove it?

Problem 2.12 ($\star$). What is $2 - 5$? Can you prove it?

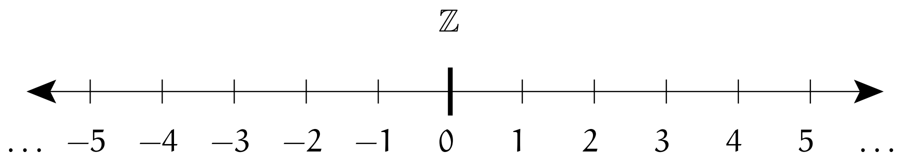
Now that we have defined negative numbers, notice that we can do away with subtraction, since subtracting n is the same as adding $(-n)$:

$$a - b = a + (-b). \tag{2.6}$$

[13] Notice that I'm intentionally vague about what sorts of numbers this definition applies to: that's because we want this definition to apply to *all* kinds of numbers, even the kinds we haven't talked about yet.

This is what I meant when I said there's "no such thing as subtraction." Of course I didn't really mean it literally. All I meant is that once we have negative numbers, we can define subtraction in terms of addition, so that subtraction is just a shorthand for a certain kind of addition rather than being a completely new operation in and of itself. Sometimes it can be helpful to think of subtraction in this way, as addition in disguise. Usually, though, no one gets too confused by subtraction, so it's not really that big of a deal. However, people *do* tend to get confused by division; we'll see in a minute that there's no such thing as division, either, which is much more helpful.

Our new set of numbers—negative numbers, zero,[14] and positive numbers—is called the set of *integers*, which can be represented by the symbol $\mathbb{Z}$.[15]



2.6 There's no such thing as division, either

Are you confused by division? Never fear, it doesn't exist! This time you probably have a much better idea of where we're going with this. Just as subtraction can be seen as adding additive inverses . . .

Problem 2.13 ($\star$). Complete the above sentence.

Division is the 'opposite' of multiplication in exactly the same way that subtraction is the opposite of addition:

$$c/b = a \iff a \cdot b = c. \tag{2.7}$$

[14]Note that zero is considered to be neither positive nor negative. Hence saying "all positive numbers" is slightly different than saying "all nonnegative numbers." The latter includes zero, but the former does not.

[15]That's right, $\mathbb{Z}$, not $\mathbb{I}$. It stands for *Zahlen* (or perhaps $\mathfrak{Zahlen}$), which is German for *numbers*. People who speak English aren't the only ones who do math, you know.

So, for example, $21/7 = 3$ since $3 \cdot 7 = 21$. As another example, $-22/11 = -2$ since $-2 \cdot 11 = -22$.[16] And as a final example, $13/6 =$ Aieeeeeeeeeeeeee!

Not again!

It should be clear by now (if you didn't already see this one coming) that the integers are not closed under division, since, for example, there is no integer which can be multiplied by 6 to get 13. We'll just have to fix things up by once more extending our idea of what counts as a number. We'll start in much the same way that we did when we introduced negative numbers.

For any integer q, we can define the *reciprocal* of q, written $1/q$, as the *multiplicative inverse of* q—that is, $1/q$ is the number for which

$$q \cdot 1/q = 1. \tag{2.8}$$

Problem 2.14 ($\star$). There is one small problem with the above definition. Can you spot it? ★

You got it, we can't quite do this for *any* integer q. In particular:

Problem 2.15 ($\star\star$). Explain why defining $1/0$ according to the definition given above doesn't make any sense.

OK, so we can define $1/q$ for any integer q *except zero*. Zero is just special like that.[17]

It's easy to see now that we can think of division as a certain kind of multiplication, just as we thought of subtraction as a certain kind of addition. In particular, dividing is the same as multiplying by a reciprocal:

$$a \div b = a \cdot 1/b \tag{2.9}$$

It's often much easier to think of division this way, in terms of multiplication by a reciprocal, especially when dealing with fractions.

[16] The astute reader might note that we never actually defined what it means to multiply with negative numbers. But actually, we have: it's possible to logically deduce what it must mean to multiply negative numbers using only things we have defined and proved so far; see Problem 2.36.

[17] Let's just put it this way, you wouldn't want to meet zero in a dark alley.

So, have we solved our problem? Not quite. Our new set of numbers (integers along with their reciprocals) is no longer closed under addition!

Problem 2.16 ($\star$). Give an example that shows why the set of integers along with their reciprocals is not closed under addition.

2.7 Rational numbers

We can define the set of *rational numbers* as all numbers of the form $\frac{p}{q}$, where p and q are both integers, and q is not zero. (We can also write $\frac{p}{q}$ as p/q.) To connect rational numbers to our previous definitions, we note that

$$\frac{p}{q} = p \cdot 1/q. \tag{2.10}$$

Problem 2.17 ($\star$). Compare equations (2.9) and (2.10). What do you conclude?

Problem 2.18 ($\star$). What is the multiplicative inverse of a rational number $\frac{p}{q}$?

Problem 2.19 ($\star$). What is

$$\frac{a}{b} \div \frac{c}{d}?$$

Can you prove it? ★

There, now you'll never be confused by division with fractions again!

You may note that we haven't actually defined how to do arithmetic with rational numbers; in fact we're not even sure that the rational numbers are closed under addition (as I claimed they would be)! In the interest of space I won't do that here—but you should feel free to prove for yourself how to do arithmetic with rational numbers, and the fact that the rational numbers are closed under addition and multiplication. You already have all the tools you need!

Instead, let's examine some amazing properties of the set of rational numbers, which, by the way, can be represented by the symbol $\mathbb{Q}$.[18]

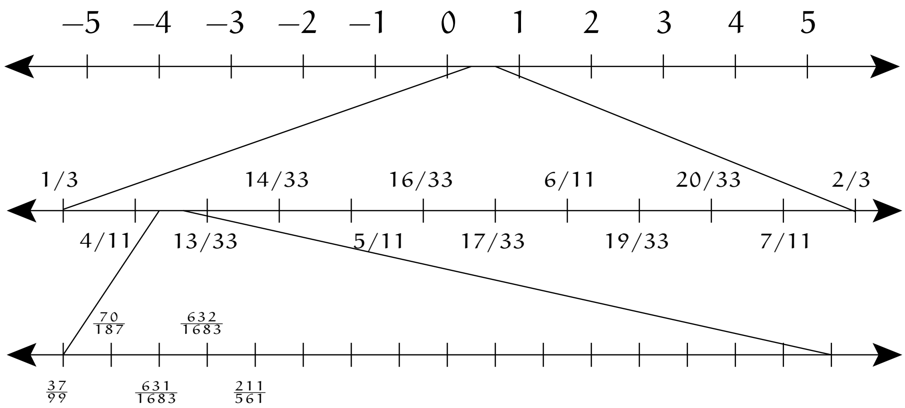
[18] No, it's not German. It stands for *quotient*.

Problem 2.20 ($\star\star$). Let ε represent a small positive number. Show that no matter how small ε is, you can always find a rational number $p/q < \varepsilon$.[19] ★

In other words, you can 'zoom in' towards zero as far as you want and you will always find smaller and smaller rational numbers no matter how close you get. But that's not the end of the story; as it turns out, there's nothing particularly special about zero here at all:

Problem 2.21 ($\star\star\star$). Given *any* number x and any small positive number ε, show that you can always find a rational number p/q which is within a distance of ε from x (that is, $|x - p/q| < \varepsilon$). ★

This is pretty wild! What this means is that you can throw a dart at the number line while blindfolded, then look at the place where the dart hit with as powerful of a microscope as you want,[20] and you'll find some rational numbers. There are rational numbers everywhere! The math term for this is that the rational numbers are *dense*. Here's an illustration to help show what's going on:



[19] That's a lowercase epsilon, by the way, which is the fifth letter in the Greek alphabet.
[20] Or as powerful as you can afford; a good scanning electron microscope can cost a few hundred thousand dollars.

2.8 Exponentiation: repeating repeatedly

When we first defined multiplication in section 2.4 We saw process of generalizing addition to multiplication, although it quickly got away from its original definition as repeated addition (e.g. 5/6 * 5/7 = 25/42? Nothing "repeated" here...). Now we'll define exponentiation as repeated mult. and see where the logic takes us.]

If x is a number and a is a positive integer, we can define

$$x^a = \underbrace{x \cdot x \cdot x \cdots x}_{a}. \tag{2.11}$$

Pretty simple, right? The weirdness starts (as anyone who has had experience with exponents knows) when we try to define what exponentiation means when the exponent is something other than a positive integer. But as we'll see, there are good ways to define exponentiation in these cases that make a lot of sense once you understand the reasons behind them. We'll start by proving some properties of exponentiation for positive integer exponents, and then see how to extend the definition of exponentiation to other sorts of exponents so that our nice properties still work. Ready?

Problem 2.22 ($\star$). Show that to multiply two expressions that have the same base to different powers, you add the exponents. That is, show that if a and b are positive integers, and x is any number,

$$x^a \cdot x^b = x^{a+b}. \qquad \bigstar \tag{2.12}$$

Problem 2.23 ($\star$). Show that if a and b are positive integers, with $a > b$, and x is any number other than zero,[21]

$$\frac{x^a}{x^b} = x^{a-b}. \tag{2.13}$$

Problem 2.24 ($\star$). Show that if a and b are positive integers,

$$(x^a)^b = x^{ab}. \qquad \bigstar \tag{2.14}$$

[21] By the way, why can't we have $x = 0$ here?

OK, now we understand how to do certain manipulations with positive integer exponents. But what on earth does something like this mean:

$$16^{-\frac{3}{4}}?$$

I know what you're thinking: it means we should run away. But wait, come back! This is no time to be a smart-aleck. . . . Actually, I take that back. This is an excellent time to be a smart-aleck. A mathematical smart-aleck would take one look at equation (2.13) and say, "$a > b$, huh? What happens if I make $a \leqslant b$?" This is exactly the kind of comment that won't get you very far with your teacher, but will get you quite far indeed in mathematics![22] Great mathematicians ask this sort of question all the time. "What happens if I break the rules? When can I get away with it? How can I change the definition so it doesn't need as many rules?" The worst that can happen if you break the rules is that it doesn't work—and even in that case, you might at least gain a better understanding of why the rules are necessary!

So, let's break the rules. Equation (2.13) says you can only use it when $a > b$. But if $a \leqslant b$ we might have something like this:

$$\frac{x^2}{x^5}.$$

Applying equation (2.13) (even though the rules say we can't!), this should be equal to $x^{2-5} = x^{-3}$. What could be simpler?

"Wait just one minute! x to the negative three!?" (you might be thinking). "See, this is why we shouldn't break the rules, because we get horrible things that don't make any sense! We only know what exponentiation means when the exponent is a positive integer. How on earth can you multiply something by itself negative three times?" Well, you have a point—but hang on just a minute. Instead of looking at this as a horrible failure, try to see it as a golden opportunity. Since we don't know what x^{-3} means, we are free to define it however we like. But of course most definitions would be silly. For example, we would be 'free' to define $x^{-3} = \pi$, but that

[22] Don't tell your teacher I said that.

wouldn't be very helpful. The golden opportunity is to define x^{-3} (and negative exponents in general) in such a way that our definition fits neatly with the properties we've already established for positive exponents.

Problem 2.25 ($\star\star$). Let a be a positive integer. We already know what x^a means. How should we define x^{-a} so that it fits neatly with the exponent properties we've already proven? ★

Now we know what to do with exponents that are positive integers or negative integers. But what about zero?

Problem 2.26 ($\star\star$). Let x be any number other than zero. What should x^0 be defined as? Why? ★

As an exercise, you can check that with our new definitions of exponentiation by zero and negative integers, equations (2.12) and (2.14) are now true for any integer exponents, not just whole numbers.

That wasn't too hard, was it? Now let's tackle rational exponents.

Problem 2.27 ($\star\star$). If we want to define exponentiation for *any* rational exponent, it is enough to define exponentiation for exponents of the form $\frac{1}{q}$, where q is a positive integer (assuming we want our definition of rational exponentiation to play nicely with our other exponentiation facts). Why? ★

Problem 2.28 ($\star\star$). Let x be any number, and let q be a positive integer. How should we define $x^{1/q}$? Why? ★

Now we're finally ready to put all of this together!

Problem 2.29 ($\star\star$). What is the value of the following expressions? ★

(a) $16^{-\frac{3}{4}}$

(b) $\left(x^{-1}\sqrt{y^3}\right)^2 - \left(\frac{\sqrt[3]{x}^{10}}{y^5}\right)^{-\frac{3}{5}}$

2.9 A diversion (repetitively repeating repeatedly)

So, multiplication is repeated addition; exponentiation is repeated multiplication; the obvious next step would be . . . repeated exponentiation! Repeated exponentiation *has* in fact been studied; sometimes arrow notation is used:

$$x \uparrow\uparrow n = \underbrace{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}_{n}. \tag{2.15}$$

In other words, $x \uparrow\uparrow n$ represents a *power tower*[23] of n copies of x. (Keep in mind that by convention, stacked exponents should be evaluated from top to bottom, not bottom to top. This makes a big difference, since, for example, if we incorrectly evaluate $2 \uparrow\uparrow 4 = 2^{2^{2^2}}$ from bottom to top we get $2^{2^{2^2}} = 4^{2^2} = 16^2 = 256$, but evaluating from top to bottom we get the correct answer, $2^{2^{2^2}} = 2^{2^4} = 2^{16} = 65536$.) This is not really a very useful operation in general. But who cares? It's fun. Let's define the *superexponential function* as follows, for $n \geqslant 1$:

$$\mathfrak{S}(n) = n \uparrow\uparrow n. \tag{2.16}$$

In other words, n superexponential is n to the n to the n to the . . . and so on, repeated n times.

Problem 2.30 ($\star$). Find the values of $\mathfrak{S}(1), \mathfrak{S}(2)$, and $\mathfrak{S}(3)$.

Surprised? Well, now let's try to evaluate $\mathfrak{S}(4)$. We have

$$\mathfrak{S}(4) = 4^{4^{4^4}} = 4^{4^{256}}.$$

We can evaluate 4^{256} on any graphing calculator[24] to find that it's approximately equal to $1.34078079299 \times 10^{154}$, which means that it has 155 digits. Let's call this number B (for Big). Computer software (*e.g.* or J) can be

[23] Power Tower would be a great name for a mathematical rock band.

[24] Or Google calculator, for that matter. Try going to google.com and typing 4^256 into the search box. (You should also try typing "the answer to life, the universe, and everyting".)

used to find the value of B exactly:

$$
\begin{aligned}
B = \ & 134078079299425970995740249982058461274793658205923933777\\
& 235614437217640300735469768018742981669034276900318581864\\
& 86050853753882811946569946433649006084096.
\end{aligned}
$$

And using an online numbers-to-words converter (Fortytwo 2005), we can express this number in English just for fun:

B = thirteen quinquagintillion four hundred seven novemquardragintillion eight hundred seven octoquardragintillion nine hundred twenty nine septquardragintillion nine hundred forty two sexquardragintillion five hundred ninety seven quinquardragintillion ninety nine quattuorquardragintillion five hundred seventy four trequardragintillion twenty four duoquardragintillion nine hundred ninety eight unquardragintillion two hundred five quardragintillion eight hundred forty six novemtrigintillion one hundred twenty seven octotrigintillion four hundred seventy nine septtrigintillion three hundred sixty five sextrigintillion eight hundred twenty quintrigintillion five hundred ninety two quattuortrigintillion three hundred ninety three tretrigintillion three hundred seventy seven duotrigintillion seven hundred twenty three untrigintillion five hundred sixty one trigintillion four hundred forty three novemvigintillion seven hundred twenty one octovigintillion seven hundred sixty four septvigintillion thirty sexvigintillion seventy three quinvigintillion five hundred forty six quattuorvigintillion nine hundred seventy six trevigintillion eight hundred one duovigintillion eight hundred seventy four unvigintillion two hundred ninety eight vigintillion one hundred sixty six novemdecillion nine hundred three octodecillion four hundred twenty seven septdecillion six hundred ninety sexdecillion thirty one quindecillion eight hundred fifty eight quattuordecillion one hundred eighty six tredecillion four hundred eighty six duodecillion fifty undecillion eight hundred fifty three decillion seven hundred fifty three nonillion eight hundred eighty two octillion eight hundred eleven septillion nine hundred forty six sexillion five hundred sixty nine quintillion nine hundred forty six quadrillion four hundred thirty three trillion six hundred forty nine billion six million eighty four thousand ninety six.

Woah.

But wait—we're not done yet! $\mathfrak{S}(4)$ isn't B, it's 4^B. That is,

$$\mathfrak{S}(4) = 4^{1.34078079299\cdots\times 10^{154}}.$$

In other words, to get $\mathfrak{S}(4)$, take the number 4 and multiply it by itself about thirteen quinquagintillion[25] times. This is an incredibly, ridiculously, fantastically, monstrously large number. To give a bit of perspective, the number of atoms in the entire universe is currently estimated at somewhere around 10^{78}—that is, a one with 78 zeros after it. $\mathfrak{S}(4)$, by contrast, has about $\log \mathfrak{S}(4) = 4^{256} \log 4 \approx 8.07230473 \times 10^{153}$ digits. This means that if you could somehow write a digit on every single atom in the entire universe,[26] you would not have anywhere near enough atoms to even *write down* $\mathfrak{S}(4)$! In fact, you would need about 10^{76} (that is, ten quattuorvigintillion, or ten thousand trillion trillion trillion trillion trillion trillion) universes in order to have enough atoms.

Before you get too excited, though, keep in mind that $\mathfrak{S}(4)$ is actually a very small number, in the sense that *most* numbers are bigger than it. For example, $\mathfrak{S}(5)$ is a LOT bigger than $\mathfrak{S}(4)$. So is $\mathfrak{S}(100)$. And $\mathfrak{S}(\mathfrak{S}(4))$. And let's not forget the hypermegaexponential function $\mathfrak{HME}(x)$, which is defined in terms of repeated superexponentiation, and the ultrahypermegaexponential function, and . . .

Don't hurt your brain.

2.10 Irrational numbers

Remember how you proved in Problem 2.21 that the rational numbers are *dense*—that no matter how far you zoom in, you can always find more rational numbers hiding in every nook and cranny? This seems like pretty good evidence that any quantity can be expressed as a rational number p/q. The ancient Greeks believed this to be true; you might have to make p and q *really big* in order for p/q to be equal to your favorite number x (the thinking went), but *eventually* you will find values for p and q that work.

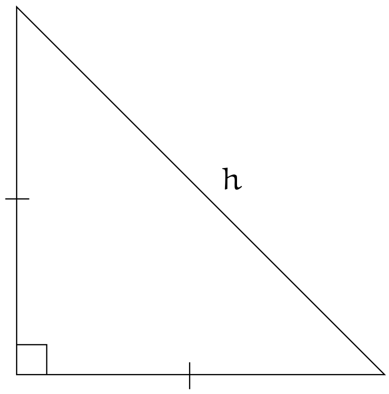
[25] Quinquagintillion is a great number to impress your friends with. Make sure they're the right sort of friends, though.

[26] A dubious proposition.

Seems reasonable, doesn't it? There are always more and more rationals no matter how far you zoom in anywhere on the number line. There are rationals everywhere! They're like mosquitos in a swamp. So the idea that every number is rational seems very reasonable indeed. There's only one teensy little problem: it's wrong.

It turns out that there are tricksy numbers (called *irrationals*) that can never be represented as a rational number p/q. No matter how far you zoom in—no matter how fine you make your rational net in an attempt to catch an irrational number—these elusive creatures will keep slipping through the cracks. Let's see if we can catch one with a net made of proof instead of rationals.[27]

Legend has it that a Greek mathematician named Hippasus made this discovery by considering a right isosceles triangle (that is, a right triangle with two equal legs):



Let's say that the legs have length 1. Obviously the hypotenuse must have a length too: let's call it h. Hippasus' amazing insight was to realize that h can never be expressed as a rational number![28]

Problem 2.31 ($\star$). Show that $h^2 = 2$. ★

[27] Be vewy, vewy quiet. We're hunting iwwationals.

[28] Actually, he most likely proved that the hypotenuse is not *commensurate* with the legs; that is, you can't find a smaller length that will evenly divide both the hypotenuse and the legs. But it really amounts to the same thing.

OK, that was the easy part. Now for the real insight! As an aside which may help you a bit, note that given any rational number p/q, we can always find a way to represent it as p'/q', where p' and q' have no common factors, and $q' > 0$. For example, $21/(-15)$ can be reexpressed as $-7/5$. So when using a rational number in a proof, it's always OK to assume that it's in lowest terms and has a positive denominator, which can sometimes make things a little easier.

Problem 2.32 ($\star\star\star$). Prove that h is irrational—that is, that h cannot be expressed as a rational number p/q. (*Hint*: use a proof by contradiction.) ★

h, of course, is the number we would call $\sqrt{2}$, or $2^{1/2}$. I hope you can appreciate what a wondrous and surprising result this is—there are rationals all over the place, packed as close together as you could ever want, and yet there are still numbers that slip through! In fact, legend has it that the Pythagoreans were so surprised and disturbed by Hippasus' proof, that they killed him. Or sacrificed a hundred bulls. Or something like that. We don't actually know exactly what they did, but the point is that this was an incredible, earth-shaking discovery.

I said "numbers" in the last paragraph, but I should point out that, technically speaking, all we *really* know at this point is that there is *at least one* irrational number. Maybe $\sqrt{2}$ is the only one! As you can probably guess, that isn't actually the case. It's not too hard to find a few more:

Problem 2.33 ($\star\star\star$). Let d be any positive integer. For which values of d is $\sqrt{d}$ rational, and for which is it irrational?

So, now we know that most square roots are irrational. Are there other irrational numbers? You bet. For example, $\sqrt[3]{2}$ is irrational. (Can you prove it?) The beloved number π, the ratio of any circle's circumference to its diameter, is also irrational (as first proven by Lambert in 1761). So is e, the base of the natural logarithm (ln); proving this is much easier than it is for π, although it requires some calculus so we won't prove it here.

How about $\pi + e$? Do you think it's rational, or irrational? The amazing thing is . . . *no one knows*! You might think that adding two irrational numbers together would surely give you another irrational, but—

Problem 2.34 ($\star\star$). If x is an irrational number and r is a rational number, prove the following: ★

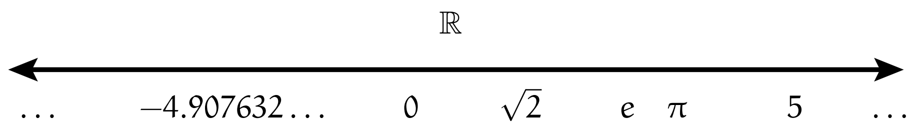
(a) $x + r$ is also irrational.
(b) $x \cdot r$ is also irrational (unless $r = 0$).

Problem 2.35 ($\star$). Use the result of Problem 2.34 to find two irrational numbers whose sum is a rational number.

So, just because π and e are irrational doesn't necessarily tell us anything about $\pi + e$. If you asked a random mathematician to guess whether $\pi + e$ is rational, they would probably tell you they think it isn't. But no one has been able to *prove* it.

It turns out that there are lots of irrational numbers; we've only scratched the surface. In fact—you should probably be sitting down for this—*most* numbers are irrational! What I mean is this: if you take the number line and pick a spot on it randomly (say, by throwing a dart with infinitely horrible aim), you have an *infinitely small chance* of hitting a rational number.[29] But wait a minute—aren't there infinitely many rational numbers?! Yup. And there are infinitely many irrational numbers, too. But the 'infinity' of the irrationals is *a bigger sort of infinity* than the 'infinity' of the rationals! Did you know there are different kinds of infinity? No? Well, as crazy as it sounds, it's true, and we'll explore this (and many other strange topics) more in Chapter ??, Infinity.

The set of all the numbers on the number line—the rational numbers together with the irrational numbers—is called the set of *real numbers*, represented by the symbol $\mathbb{R}$.



There are no hash marks on the number line shown above to emphasize the *continuity* of the real numbers. Unlike with the integers and the rationals,

[29] Of course, this immediately suggests several foolproof betting scams . . .

there are no 'gaps' in between the real numbers. So, have we completed our quest to catalog numbers? Well . . . not really.

2.11 . . . and beyond

You may not realize it, but right now we're actually in a bit of a fix. We defined exponentiation for rational numbers, which is great because it lets us talk about numbers like $\sqrt{2}$. But there's one major problem: the real numbers aren't closed under this operation! $\sqrt{2}$ is well and good, but what about $\sqrt{-64}$? Whenever you square a real number, you get a *nonnegative* real number, so there isn't any real number which is the square root of -64, or for that matter, of any negative real number at all. Aieeeeeeeeeeeeeee!

Our problems aren't limited to square roots, either; any rational power with an even denominator causes trouble (fourth roots, sixth roots . . .). Fixing this problem brings us into the world of *complex numbers* . . . but that will have to wait for another chapter.

2.12 More problems

Problem 2.36 (⋆⋆⋆). Prove the following law that defines multiplication by negative numbers: ★

$$(-a) \cdot b = -(a \cdot b). \tag{2.17}$$

Problem 2.37 (⋆⋆). Given equation (2.17), prove that

$$(-a) \cdot (-b) = a \cdot b. \tag{2.18}$$

Problem 2.38 (⋆⋆). Prove that $\log_{10}(2)$ is irrational. ★

Hints for Chapter 2

2.1 Come on, you can do better than *that*.

2.2 You can do it a lot faster if you remember that addition is commutative . . .

2.4 There are many possibilities. An informal proof might involve arranging pennies in the shape of a rectangle. A formal proof might use the definition of multiplication.

2.7 Use the definition of multiplication along with the associative and commutative properties of addition.

2.8 Note that to *disprove* a statement like this, all you have to do is show a single example where addition does not distribute over multiplication.

2.10 Start by writing 5 as $2 + 3$.

2.14 *Any* integer q, eh?

2.19 Use the result from Problem 2.18 along with equation (2.9).

2.20 Let $Q = \lceil 1/\varepsilon \rceil + 1$. What can you say about Q? ($\lceil x \rceil$ means 'round up.')

2.21 First note that you can always find a rational number $r/s < \varepsilon$, by Problem 2.20. How can you use this number to create another rational number that must be within a distance of ε from x?

2.22 Write out x^a and x^b as a product of a certain number of copies of x, according to the definition of exponentiation.

2.24 Again, use the definition of exponentiation—you can start by writing $(x^a)^b$ as the product of a certain number of copies of x^a.

2.25 Consider equation (2.13). How can we make it true when $a < b$? For example, what should x^{-3} be defined as if it is going to equal $\frac{x^2}{x^5}$?

2.26 Consider equation (2.13) when $a = b$.

2.27 Let p/q be any rational number with $q > 0$ (p could be negative), and rewrite $x^{p/q}$ in a form involving exponentiation by $1/q$.

2.28 Consider $(x^{1/q})^q$ and apply equation (2.14).

2.29 (a) First rewrite it as $16^{-\frac{3}{4}} = (16^{1/4})^{-3}$.

2.31 Use the Pythagorean Theorem; Hippasus and Pythagoras were actually friends.

2.32 There are actually quite a lot of ways to prove this, but one way (used by the ancient Greeks) might run along these lines: start by assuming that $h = \frac{p}{q}$, where p and q are integers with no common factors. Square both sides and multiply by q^2. Conclude that p must be even (why?), and substitute $2r$ for p. Now what can you conclude about q? How does this contradict your initial assumptions?

2.34 Use a proof by contradiction. For instance, what if $x + r$ were a rational number?

2.36 Start by noting that $0 = 0 \cdot b$, then expand the 0 on the right-hand side into something more useful . . .

2.38 Let $x = \log_{10}(2)$ and deduce that $10^x = 2$. Now assume that $x = p/q$ and arrive at a contradiction.

Bibliography

Conway, John H., and Richard K. Guy (1996). *The Book of Numbers*. New York: Copernicus.

Fortytwo, Geoff (2005). *BigNumbers*. 2005. URL: http://g42.org/tiki/tiki-index.php?page=BigNumbers.

Solutions

Chapter 2

2.1 Of course, this question is impossible to answer, as hard as you might try—as soon as you think of a number n you can always think of $n + 1$!

2.2 Since addition is commutative and associative, when adding up a bunch of numbers we can rearrange them in any order we want:

$$\begin{aligned} & 2+3+8+7+6+4+5+9+5+1+23+7 \\ = \; & 2+8+3+7+6+4+5+5+9+1+23+7 \\ = \; & 10+10+10+10+10+30 \\ = \; & 80. \end{aligned}$$

2.3 Following the same pattern we saw in defining zero as the identity with respect to addition, the identity with respect to multiplication (call it j) must be the number which has no effect when multiplying, or more mathematically, the number for which $a \cdot j = j \cdot a = a$ for any number a. Clearly $j = 1$.

2.4

$$\begin{aligned}
a \cdot b &= \underbrace{b + b + b + \cdots + b}_{a} \\
&= a + \underbrace{(b-1) + (b-1) + \cdots + (b-1)}_{a} \\
&= a + a + \underbrace{(b-2) + (b-2) + \cdots + (b-2)}_{a} \\
&= \underbrace{a + a + a + \cdots + a}_{n} + \underbrace{(b-n) + (b-n) + \cdots + (b-n)}_{a} \\
&= \underbrace{a + a + a + \cdots + a}_{b} \\
&= b \cdot a.
\end{aligned}$$

2.5 A formal proof using the definition of multiplication is possible. However, it's probably a lot easier (if less formal) to note that $a \cdot (b \cdot c)$ corresponds to the volume of a rectangular solid with dimensions a, b, and c, and that the volume of the solid stays the same no matter which way you turn it.

2.6 Multiplication of natural numbers is defined as repeated addition of natural numbers; therefore, if the natural numbers are closed under addition, they must be closed under multiplication, too.

2.7
$$\begin{aligned}
a \cdot (b + c) &= \underbrace{(b+c) + (b+c) + \cdots + (b+c)}_{a} && \text{(defn. of } \cdot) \\
&= \underbrace{b + b + \cdots + b}_{a} + \underbrace{c + c + \cdots + c}_{a} && \text{(properties of } +) \\
&= a \cdot b + a \cdot c.
\end{aligned}$$

2.8 Addition does not distribute over multiplication. You can find particular instances where it does; for example, $0 + (2 \cdot 3) = (0 + 2) \cdot (0 + 3)$, and $3 + (2 \cdot (-4)) = (3 + 2) \cdot (3 - 4)$. But it is also easy to find instances where it doesn't; for example, $3 + (1 \cdot 4) \neq (3 + 1) \cdot (3 + 4)$.

2.9 By definition, -0 is the number such that $-0 + 0 = 0$. But since $a + 0 = a$ for all numbers a, we know that $-0 + 0 = -0$ as well. Hence $-0 = 0$.

To see that $-1 \neq 1$, use a proof by contradiction. If $-1 = 1$, then we can add 1 to both sides to get $-1 + 1 = 1 + 1$, which simplifies to the preposterous equation $0 = 2$; therefore $-1 \neq 1$.

2.10 $5 + (-3) = (2 + 3) + (-3) = 2 + 0 = 2$.

2.11 Since -7 is the additive inverse of 7, we know that $7 + (-7) = 0$. But since addition is commutative, this means that $(-7) + 7 = 0$ as well; therefore 7 is the additive inverse of -7. As an aside, it should be easy to see how to extend this to a proof that $-(-a) = a$ for any number a.

2.12 If $2 - 5 = a$, then $a + 5 = 2$; the number a with this property is $a = -3$, as you proved in Problem 2.10.

2.13 " . . . division can be seen as multiplying multiplicative inverses."

2.14 The definition does not make sense when $q = 0$ (see Problem 2.15).

2.15 According to the definition, we should have $0 \cdot (1/0) = 1$. But zero times anything is zero, not one.

2.16 There are many possible solutions; for example, $1/3$ is the reciprocal of 3, but $1/3 + 1/3$ is neither an integer nor the reciprocal of an integer.

2.17 Equation (2.9) says that $a \div b = a \cdot 1/b$, and equation (2.10) says that $a \cdot 1/b = \frac{a}{b}$; so we conclude that $a \div b = \frac{a}{b}$, that is, the rational number a/b represents the result of dividing a by b.

2.18 The multiplicative inverse of $\frac{p}{q}$ is $\frac{q}{p}$, since

$$\frac{p}{q} \cdot \frac{q}{p} = p \cdot \frac{1}{q} \cdot q \cdot \frac{1}{p} = p \cdot 1 \cdot \frac{1}{p} = 1.$$

2.19 Since division is the same as multiplication by a reciprocal, we have

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

2.20 Given ε, let $Q = \lceil 1/\varepsilon \rceil + 1$, where $\lceil x \rceil$ means to 'round up' (in other words, $\lceil x \rceil$ denotes the smallest integer which is greater than or equal to x).[1] Then $1/Q < \varepsilon$, since

$$\frac{1}{Q} = \frac{1}{\lceil 1/\varepsilon \rceil + 1} < \frac{1}{\lceil 1/\varepsilon \rceil} \leqslant \frac{1}{1/\varepsilon} = \varepsilon.$$

[1] Bet you can't guess what $\lfloor x \rfloor$ means!

(Those inequalities are true since making the denominator of a fraction smaller always makes the whole fraction bigger.) For example, if $\varepsilon = 0.3$, let $Q = \lceil 1/0.3 \rceil + 1 = \lceil 3.333333\ldots \rceil + 1 = 5$. Then $1/5 < 0.3$. In general, $1/Q$ is a rational number with the desired property.

2.21 By the previous problem, we know we can find a rational number $r/s < \varepsilon$. We will show that there must be some multiple of r/s which is within a distance of ε from x; that is, we can find an integer k such that $x - \varepsilon < k \cdot \frac{r}{s} < x + \varepsilon$. Here's the intuition: since r/s is smaller than ε, if we start with 0 and repeatedly add r/s, we're eventually going to find a multiple of r/s within ε of x—we can't possibly 'skip over it' since r/s is too small. We can prove this more formally by contradiction: suppose that there is no such multiple of r/s. Then let j be the largest integer for which $j \cdot \frac{r}{s} \leqslant x - \varepsilon$. By our assumption, no multiples of r/s are within ε of x, so the next multiple after $j \cdot \frac{r}{s}$ must be greater than or equal to $x + \varepsilon$; that is, $(j+1) \cdot \frac{r}{s} \geqslant x + \varepsilon$. But then $(j+1) \cdot \frac{r}{s} - j \cdot \frac{r}{s} = \frac{r}{s} \geqslant 2\varepsilon$, a contradiction since we chose $r/s < \varepsilon$. Therefore such a k must exist, and in fact $k \cdot \frac{r}{s} = \frac{kr}{s}$ is a rational number within a distance of ε from x, as desired.[2]

2.22

$$\begin{aligned} x^a \cdot x^b &= (\underbrace{x \cdot x \cdots x}_{a}) \cdot (\underbrace{x \cdot x \cdots x}_{b}) \\ &= \underbrace{x \cdot x \cdots x}_{a+b} \\ &= x^{a+b}. \end{aligned}$$

2.23 Since $a > b$, we can cancel the b copies of x in the denominator with b of the x's in the numerator, leaving us with $a - b$ copies of x in the numerator, that is, x^{a-b}.

2.24

[2] It is interesting to note that this is an *existence proof*: the proof only shows that such a multiple of r/s exists, not how to find it (although finding an appropriate multiple is not, in fact, difficult).

$$
\begin{aligned}
(x^a)^b &= \underbrace{x^a \cdot x^a \cdot x^a \cdots x^a}_{b} \\
&= \overbrace{(\underbrace{x \cdots x}_{a}) \cdots (\underbrace{x \cdots x}_{a})}^{b} \\
&= \underbrace{x \cdots x}_{ab} = x^{ab}.
\end{aligned}
$$

2.25 If a is a positive integer, we should define

$$x^{-a} = \frac{1}{x^a}.$$

2.26 We want equation (2.13) to work when $a = b$, so we should define

$$x^0 = x^{a-a} = \frac{x^a}{x^a} = 1.$$

Another way to think about it is that $x^0 \cdot x^1 = x^0 \cdot x$ should be equal to $x^{0+1} = x^1 = x$; so if $x^0 \cdot x = x$, x^0 must be equal to 1. Note that 0^0 poses a problem: should it be equal to 1 (according to the 'rule' that anything to the zero power is 1), or 0 (according to the 'rule' that zero to any power is 0), or should it be undefined? In the end, it depends on your perspective. In the context of calculus, 0^0 corresponds to an *indeterminate form*

2.27 Let p/q be any rational number; we can assume $q > 0$. According to equation (2.14), we have

$$x^{\frac{p}{q}} = x^{\frac{1}{q} \cdot p} = (x^{\frac{1}{q}})^p.$$

But p is an integer, and we already know how to exponentiate by any integer; so as long as we define $x^{1/q}$ we know how to do the rest.

2.28 We should define

$$x^{1/q} = \sqrt[q]{x},$$

since that means

$$x = x^{q/q} = (x^{1/q})^q = (\sqrt[q]{x})^q = x.$$

2.29 (a) $16^{-\frac{3}{4}} = \left(16^{1/4}\right)^{-3} = (\sqrt[4]{16})^{-3} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$

(b) This one is a little more complicated-looking but yields to patient application of exponent identities:

$$\begin{aligned}\left(x^{-1}\sqrt{y^3}\right)^2 - \left(\frac{\sqrt[3]{x}^{10}}{y^5}\right)^{-\frac{3}{5}} &= \left(x^{-1}y^{\frac{3}{2}}\right)^2 - \left(\frac{x^{\frac{10}{3}}}{y^5}\right)^{-\frac{3}{5}} \\ &= x^{-1\cdot 2}y^{\frac{3}{2}\cdot 2} - \frac{x^{\frac{10}{3}\cdot(-\frac{3}{5})}}{y^{5\cdot(-\frac{3}{5})}} \\ &= x^{-2}y^3 - \frac{x^{-2}}{y^{-3}} \\ &= \frac{y^3}{x^2} - \frac{y^3}{x^2} = 0.\end{aligned}$$

2.30 $\mathfrak{S}(1) = 1$; $\mathfrak{S}(2) = 2^2 = 4$; $\mathfrak{S}(3) = 3^{3^3} = 3^{27} = 7625597484987$ (that's 7.6 trillion, slightly smaller than the US national debt).

2.31 By the Pythagorean Theorem, $h^2 = 1^2 + 1^2 = 2$.

2.32 Suppose $h = \frac{p}{q}$, where $\frac{p}{q}$ is in lowest terms. Squaring both sides and multiplying by q^2 gives us

$$q^2h^2 = p^2,$$

and since we know $h^2 = 2$ we can write

$$2q^2 = p^2.$$

This means that p^2 must be even since it is equal to 2 times something; but that means p itself must be even since squaring an odd number always gives another odd number. So we can substitute $2r$ for p:

$$2q^2 = (2r)^2.$$

Simplifying a bit yields

$$q^2 = 2r^2,$$

from which we can conclude that q is also even—but that's a contradiction since we assumed p and q don't have any common factors! Therefore our initial assumption must be false, and h is not rational.

2.34

(a) Suppose $x + r$ were rational. Since r is rational, we know that $-r$ is also rational; therefore, since the rationals are closed under addition, $(x + r) + (-r) = x$ must be rational as well. But we are given that x is irrational. Therefore, by contradiction, $x + r$ must be irrational as well.

(b) The proof for $x \cdot r$ is similar.

2.35 There are many possibilities; here's one. Since π is irrational we know that $(-1) \cdot \pi = -\pi$ is also irrational (by part (b) of Problem 2.34), so $3 + (-\pi) = 3 - \pi$ is irrational too (by part (a)). But then we see that π and $3 - \pi$ are two irrational numbers which sum to a rational number (namely, 3).

2.36

$$\begin{aligned} 0 &= 0 \cdot b && \text{(by equation (2.2))} \\ &= (a + (-a)) \cdot b && \text{(by equation (2.5))} \\ &= a \cdot b + (-a) \cdot b && \text{(by equation (2.3))} \end{aligned}$$

But if $a \cdot b + (-a) \cdot b = 0$, then by equation (2.5) we know that $(-a) \cdot b$ is the additive inverse of $a \cdot b$; that is, $(-a) \cdot b = -(a \cdot b)$.

2.37

$$\begin{aligned} (-a) \cdot (-b) &= -(a \cdot (-b)) \\ &= -((-b) \cdot a) \\ &= -(-(b \cdot a)) \\ &= b \cdot a = a \cdot b. \end{aligned}$$

2.38 Let $x = \log_{10}(2)$. Then by the definition of logarithms, we know that $10^x = 2$. Suppose $x = p/q$ is rational (as usual we assume that $q \neq 0$; we also know that p and q are both positive, since 10 to a negative power would be smaller than 1). Then $10^{p/q} = 2$, and raising both sides to the q power we find that $10^p = 2^q$. But since p and q are positive integers, this is impossible: the left side is divisible by 5 but the right side is not. Therefore, by contradiction, $\log_{10}(2)$ must be irrational.